The oscillating quantaties are assume to behave sinusoidally $(e^{i(kx-\omega t)})$, so that the time derivative $\frac{d}{dt}$ can be replace by $(-i\omega)$, ~~the gradient~~ $\frac{\partial}{\partial x}$ can be replace by ik and if we put

$$\frac{eB_0}{m_e c} = \Omega_e$$

The above set of equations become

$$-i\omega n_{e1} + n_0 ik v_{e1x} = 0 \quad \text{(12)}$$

$$-i\omega v_{e1x} = -\frac{e}{m_e}E_{1x} - \frac{e}{m_e c} v_{e1y} B_0$$

$$= -\frac{e}{m_e}E_{1x} - \Omega_e v_{e1y} \quad \text{(13)}$$

$$-i\omega v_{e1y} = v_{e1x}\,\Omega_e \quad \text{(14)}$$

$$-i\omega v_{e1z} = 0 \quad \text{(15)}$$

$$ikE_{1x} = -4\pi e n_{e1} \quad \text{(16)}$$

now from (13) we get

$$v_{e1x} = \left(-\frac{e}{m_e}\right)\left(\frac{E_{1x}}{-i\omega}\right) + \frac{\Omega_e v_{e1y}}{(i\omega)}$$

$$= \frac{eE_{1x}}{im_e\omega} + \cancel{\Omega_e v_{e1y}}$$

$$= \left(\frac{e}{im_e\omega}\right)\left(-\frac{4\pi e n_{e1}}{ik}\right) + \frac{\Omega_e}{i\omega}\left(\frac{\Omega_e}{-i\omega}v_{e1x}\right)$$

$$v_{e1x} = \frac{4\pi e^2 n_{e1}}{m_e\omega k} + \frac{\Omega_e^2 v_{e1x}}{\omega^2} \quad (\text{from (14) \& (16)})$$

$$\Rightarrow \left(1 - \frac{\Omega_e^2}{\omega^2}\right) v_{e1x} = \frac{4\pi e^2}{m_e\omega k}\left(\frac{n_0 ik v_{e1x}}{i\omega}\right)$$

(from (12) n_{e1})

$$\left(1-\frac{\Omega_e^2}{\omega^2}\right)v_{e1x} = \frac{4\pi e^2 n_0}{m_e\,\omega^2}\,v_{e1x}$$

$$= \frac{\omega_{pe}^2}{\omega^2}\,v_{e1x}$$

$\left[\because \omega_{pe}^2 = \frac{4\pi e^2 n_0}{m_e}\right]$

$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}$

Electron Plasma frequency.

$$\Rightarrow (\omega^2 - \Omega_e^2) = \omega_{pe}^2$$

$$\Rightarrow \omega^2 = \omega_{pe}^2 + \Omega_e^2$$

$$= \omega_h^2$$

The frequency ω_h^2 is called upper hybrid frequency.

Electrostatic wave in the presence of magnetic field due to ion motion :-

To derive an expression for the electrostatic wave in the presence of $\vec{B}$-field due to ion motion, let us assume the following assumptions.

(i) There is static magnetic field $\vec{B_0}$ in the direction $\perp$ to the direction of the electrostatic wave.

(ii) There is no thermal motion that is $K_BT_i = 0$

(iii) The plasma is infinite in extend.

(iv) The ion oscillations occur only in the x-direction.

iv) The plasma is neutral an at rest. That is $\vec{v}_0 = 0$ $\vec{E}_0 = 0$

From the above set of ~~eqn~~ assumptions we have the ion eqn of the motion are —

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\vec{v}_i) = 0 \quad \text{—— (1)}$$

$$\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i\cdot\vec{\nabla})\vec{v}_i = -\frac{e}{m_i}\vec{\nabla}\phi$$

$$+ \frac{e}{m_i c}(\vec{v}_i \times \vec{B}) \quad \text{—— (2)}$$

We assume electrostatic waves with $\vec{K}\times\vec{E} = 0$ so that $\vec{E} = -\vec{\nabla}\phi$ and the Boltzmann ~~equation~~ relation

$$n_i = n_0 \exp\left(\frac{e\phi}{\gamma K_B T_e}\right) \quad \text{—— (3)}$$

eqn (1), (2) & (3) can be easily solve by the procedure of linearization. Then we have

$$\left.\begin{aligned} \vec{B} &= \vec{B}_0 + \vec{B}_1 \\ \vec{E} &= \vec{E}_1 \\ \phi &= \phi_1 \\ n_i &= n_0 + n_{i1} \\ \vec{v}_i &= \vec{v}_{i1} \end{aligned}\right\} \quad \text{—— (A)}$$

using (A) in eqns (1), (2) & (3) we get

$$\frac{\partial n_{i1}}{\partial t} + n_0\vec{\nabla}\cdot\vec{v}_{i1} = 0 \quad \text{—— (4)}$$

$$\frac{\partial \vec{v}_{i1}}{\partial t} = -\frac{e}{m_i}\vec{\nabla}\phi_1 + \frac{e}{m_i c}(\vec{v}_{i1}\times\vec{B}_0) \quad \text{—— (5)}$$

$$n_i = n_0 \frac{e\phi_1}{\gamma K_B T_e} \quad \text{—— (6)}$$

We shall consider only longitudinal waves $\vec{k} \parallel \vec{E}_1$, without loss of generality. We can choose the x-axis to lie along $\vec{k}$ & $\vec{E}_1$ and the z-axis to lie along $\vec{B}_0$. Thus we have

$$k_y = k_z = E_y = E_z = 0.$$

$$\vec{k} = k\hat{x}, \quad \vec{E}_1 = E_1\hat{x}$$

So writing equations (4) to (6) componentwise we get

$$\frac{\partial v_{i1x}}{\partial t} = -\frac{e}{m_i}\frac{\partial \phi_1}{\partial x} + \frac{e}{m_i c} v_{i1y} B_0$$

$$\frac{\partial v_{i1y}}{\partial t} = -\frac{e}{m_i c} v_{i1x} B_0 \quad \text{—— (7)}$$

$$\frac{\partial n_{i1}}{\partial t} + n_0 \frac{\partial v_{i1x}}{\partial x} = 0 \quad \text{—— (8)} \quad \text{—— (9)}$$

equation (6) can be written in linearized form as

$$n_{i1} = n_0 \frac{e\phi_1}{\gamma k_B T_e}$$

Considering the plasma approximation $n_{e1} = n_{i1}$ we have

$$n_{e1} = n_{i1} = n_0 \frac{e\phi_1}{\gamma k_B T_e} \quad \text{—— (10)}$$

Using the plane wave solution the time derivative $\frac{\partial}{\partial t}$ can be replaced by $-i\omega$, $\frac{\partial}{\partial x}$ can be replaced by ik and putting $\Omega_i = \frac{eB_0}{m_i c}$

Then the above set of eqns become

$$-i\omega v_{i1x} = -\frac{e}{m_i}(ik)\phi_1 + \Omega_i v_{i1y} \quad (11)$$

$$-i\omega v_{i1y} = -\Omega_i v_{i1x} \quad (12)$$

$$-i\omega n_{i1} + n_0 (ik) v_{i1x} = 0 \quad (13)$$

$$n_{i1} = n_0 \frac{e\phi_1}{\gamma k_B T_e} \quad (14)$$

From (11) we have

$$-i\omega \left[\frac{-i\omega v_{i1y}}{-\Omega_i}\right] = \cancel{-\frac{e}{m_i}(ik)\phi_1 + \Omega_i v_{i1y}}$$

$$-i\omega v_{i1x} = -\frac{e}{m_i}(ik)\phi_1 + \Omega_i\left(\frac{\Omega_i v_{i1x}}{i\omega}\right)$$

$$\Rightarrow v_{i1x} = \frac{e}{m_i}\frac{k}{\omega}\phi_1 + \frac{\Omega_i^2 v_{i1x}}{\omega^2}$$

$$\Rightarrow v_{i1x} = \frac{e}{m_i}\frac{k}{\omega}\frac{n_{i1}}{n_0}(\gamma k_B T_e) + \frac{\Omega_i^2 v_{i1x}}{\omega^2}$$

$$\Rightarrow v_{i1x} = \frac{e}{m_i}\frac{k}{\omega}\frac{\gamma k_B T_e}{n_0}\frac{ik\, v_{i1x}}{i\omega} n_0 + \frac{\Omega_i^2 v_{i1x}}{\omega^2}$$

$$v_{i1x} = \frac{\gamma k_B T_e}{m_i}\frac{k^2}{\omega^2} v_{i1x} + \frac{\Omega_i^2}{\omega^2} v_{i1x}$$

$$\Rightarrow 1 = v_{Th}^2 \frac{k^2}{\omega^2} + \frac{\Omega_i^2}{\omega^2}$$

$$\Rightarrow \omega^2 = k^2 v_{Th}^2 + \Omega_i^2 \quad \text{where} \quad v_{Th} = \sqrt{\frac{\gamma k_B T_e}{m_i}}$$

This is the dispersion relation for electrostatic ion cyclotron waves,

The lower Hybrid frequency:-

To get the expression for lower Hybrid frequency we make the following assumption.

(up previous 12 eqn)

i) There is static magnetic field B_0 in the direction $\perp^r$ to the direction of the electrostatic wave.

ii) There is no thermal motion that $KT_i = KT_e = 0$

iii) The plasma is infinitely in extent.

iv) The electron ion motion occure only in the x-direction

v) The plasma is neutral at rest that is $v_0 = 0$ and $E_0 = 0$.

$\boxed{\text{Previous Q upto 12 eqns}}$

from (11) we get

$$-i\omega v_{i1x} = -\frac{e}{m_i}(ik\phi_1) + \Omega_i\left(\frac{+\Omega_i v_{i1x}}{i\omega}\right)$$

$$\Rightarrow v_{i1x} = \frac{ek\phi_1}{m_i\omega} + \frac{\Omega_i^2}{\omega^2} v_{i1x}$$

(using (12))